

FINAL EXAMINATION

MA 109C: INTRODUCTION TO GEOMETRY AND TOPOLOGY

Questions

- (1) (25 points)
 - (a) (6 points) State the definition of an orientation of a finite-dimensional vector space, and the definition of an orientation of a manifold. (Don't forget the 0-dimensional case!)
 - (b) (7 points) Let X , Y , and Z be oriented manifolds without boundary, with Z being a submanifold of Y . Let $f : X \rightarrow Y$ be a smooth map transversal to Z . State the definition of the preimage orientation of $f^{-1}(Z)$.
 - (c) (6 points) For a non-negative integer p , state the definition of an alternating p -tensor on a vector space, and the definition of a p -form on a manifold.
 - (d) (6 points) State the (generalized) Stokes theorem.

- (2) (25 points)
- (a) (6 points) State the definition of a smooth vector field v on a manifold X , and the definition of the index of v at an isolated zero $x \in X$ of v .
 - (b) (6 points) Let X be a compact oriented manifold without boundary. State the definition of the Euler characteristic $\chi(X)$ of X . State the Poincaré–Hopf theorem for X .
 - (c) (7 points) Prove that for positive integers $k \geq 1$,

$$\chi(\mathbb{S}^k) = \begin{cases} 2, & k \text{ even} \\ 0, & k \text{ odd.} \end{cases}$$

- (d) (6 points) What is $\chi(O(7))$, where $O(7)$ is the orthogonal group consisting of the 7×7 real orthogonal matrices?

- (3) (25 points) In this problem, we will prove that there exist real numbers $x, y, z, w \in \mathbb{R}$ such that

$$\begin{cases} x^3 - 3xy^2 - 3xz^2 - 3xw^2 - 362x^2 - 12 = 0, \\ 3x^2y - y^3 - yz^2 - yw^2 + 645x - 746525w = 0, \\ 3x^2z - y^2z - z^3 - zw^2 + 14xy = 0, \\ 3x^2w - y^2w - z^2w - w^3 - 1 = 0. \end{cases} \quad (1)$$

The strategy of the proof will be to imitate the proof of the Fundamental Theorem of Algebra.

- (a) (5 points) State the Fundamental Theorem of Algebra.

Following the proof of the Fundamental Theorem of Algebra, write the above system (1) of equations as

$$F(x, y, z, w) + G(x, y, z, w) = 0,$$

where $F, G : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ are the polynomials such that each component of F is homogeneous of degree 3 and each component of G has degree less than 3. It follows from a computation, which you are not required to show, that

$$\begin{aligned} & (x^3 - 3xy^2 - 3xz^2 - 3xw^2)^2 + (3x^2y - y^3 - yz^2 - yw^2)^2 \\ & + (3x^2z - y^2z - z^3 - zw^2)^2 + (3x^2w - y^2w - z^2w - w^3)^2 \\ & = (x^2 + y^2 + z^2 + w^2)^3, \quad (x, y, z, w) \in \mathbb{R}^4, \end{aligned}$$

or equivalently,

$$|F(x, y, z, w)| = |(x, y, z, w)|^3, \quad (x, y, z, w) \in \mathbb{R}^4,$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^4$.

- (b) (5 points) Show that there exists some $R > 0$ such that

$$|F(x, y, z, w)| > |G(x, y, z, w)|$$

whenever $(x, y, z, w) \in \mathbb{R}^4$ is such that $|(x, y, z, w)| = R$.

Now, assume for contradiction that there are no solutions to (1) in $\mathbb{R}^4$.

- (c) (8 points) For $R > 0$ given as in (b), show that the map $F/|F| : \partial B_R(0) \rightarrow \mathbb{S}^3$ is well-defined and has degree zero.
 (d) (7 points) Use the Borsuk–Ulam theorem on $F/|F| : \partial B_R(0) \rightarrow \mathbb{S}^3$ to deduce that the mod 2 degree of $F/|F|$ is 1. Show that its degree must be odd.

The contradiction between (c) and (d) tells us that (1) must have a solution, as we set out to prove.

(4) (25 points)

- (a) (8 points) Let m be a positive integer. Consider the unit circle $\mathbb{S}^1$ as a subset of $\mathbb{C}$, given the boundary orientation as the boundary of the unit disc $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$. Define the map $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ as $f(z) = z^m$. Show that $1 \in \mathbb{S}^1$ is a regular value of f with exactly m preimages, each point at which the differential df is orientation preserving. From this deduce that the degree of f is m .
- (b) (8 points) Now, considering $\mathbb{S}^1$ as a subset of $\mathbb{R}^2$ with coordinates (x, y) , consider the differential 1-form $\omega = -ydx + xdy$ (more precisely, the 1-form $-ydx + xdy$ is defined on $\mathbb{R}^2$, and we pull this back to $\mathbb{S}^1$ via the inclusion map). Compute the integral

$$\int_{\mathbb{S}^1} \omega = 2\pi$$

and the pullback

$$f^*\omega = m\omega.$$

Thus

$$\int_{\mathbb{S}^1} f^*\omega = m \int_{\mathbb{S}^1} \omega \quad \text{and} \quad \int_{\mathbb{S}^1} \omega \neq 0.$$

Explain why this and the degree formula gives a different proof that the degree of f is m .

(Hint: for computing the pullback, note that if we define the vector field $v(x, y) = (-y, x)$ on $\mathbb{S}^1$, then $\frac{d}{dz}z^m = mz^{m-1}$ for z a complex variable implies that $df(v) = mv$. Now observe that $\omega(v) = 1$.)

- (c) (9 points) Now consider the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$, given the boundary orientation from the closed unit ball. Define the map $g : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ given by

$$g(x, y, z) = \left(\frac{x^2 - y^2}{1 + z^2}, \frac{2xy}{1 + z^2}, \frac{2z}{1 + z^2} \right), \quad (x, y, z) \in \mathbb{S}^2.$$

Check that this map is well-defined and smooth. Consider the differential 2-form on $\mathbb{S}^2$ given by $\nu = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$ (more precisely, this 2-form is defined on $\mathbb{R}^3$, and we pull this back to $\mathbb{S}^2$ via the inclusion map). Compute the integral

$$\int_{\mathbb{S}^2} \nu = 4\pi$$

and the pullback

$$g^*\nu = \frac{4(1 - z^2)}{(1 + z^2)^3} (2zdx \wedge dy + xdy \wedge dz + ydz \wedge dx).$$

Thus compute

$$\int_{\mathbb{S}^2} g^*\nu = 8\pi.$$

Explain why this and the degree formula shows that $\deg(g) = 2$.¹

¹For reference, the map g is the map induced by the square map $z \mapsto z^2 : \mathbb{C} \rightarrow \mathbb{C}$ via stereographic projection. In general, any rational function $P(z)/Q(z)$ induces a smooth map from the Riemann sphere $\mathbb{S}^2$ to itself, and the degree of that map is $\deg P - \deg Q$. You are not required to show any of the statements in this footnote.

Hint: you may use the fact that

$$\int_{\mathbb{S}^2} (adx \wedge dy + bdy \wedge dz + cdz \wedge dx) = \int_{\mathbb{S}^2} (ax + by + cz)dS,$$

where dS denotes the area element on $\mathbb{S}^2$. Also, note that

$$\frac{d}{dz} \left(\frac{z}{1+z^2} \right) = \frac{1-z^2}{(1+z^2)^2}.$$