MIDTERM

MA 109C: INTRODUCTION TO GEOMETRY AND TOPOLOGY

Questions

(1) (a) (8 points) Given sets X and Y a function $f : X \to Y$, the *graph* of f is defined as

$$\mathrm{graph}(f) := \{(x, f(x)) \in X \times Y : x \in X\}.$$

Define the function $F : X \to \mathrm{graph}(f)$ by

$$F(x) = (x, f(x)), \quad x \in X.$$

Show that if X and Y are manifolds and f is smooth, then $\mathrm{graph}(f)$ is a manifold and $F : X \to \mathrm{graph}(f)$ is a diffeomorphism. Show that F has derivative

$$dF_x(v) = (v, df_x(v)), \quad x \in X,\ v \in T_x(X),$$

when we view $\mathrm{graph}(f)$ as a submanifold of the product manifold $X \times Y$. Show that

$$T_{(x,f(x))}(\mathrm{graph}(f)) = \mathrm{graph}(df_x : T_x(X) \to T_{f(x)}(Y)).$$

(b) (5 points) Given a set X, the diagonal $\Delta(X)$ is defined as

$$\Delta(X) := \{(x, x) \in X \times X : x \in X\}.$$

Show that if X is a manifold, then $\Delta(X)$ is a manifold of the same dimension as X. Show that

$$T_{(x,x)}(\Delta(X)) = \Delta(T_x(X)).$$

(c) (5 points) Let V be a finite dimensional real vector space, and let $A : V \to V$ be a linear map. Show that, as submanifolds of $V \times V$, $\mathrm{graph}(A) \pitchfork \Delta(V)$ if and only if 1 is not an eigenvalue of A.

(d) (7 points) Let X be a manifold and let $f : X \to X$ be a smooth map. A fixed point $x \in X$ of f, i.e., a point $x \in X$ such that $f(x) = x$, is a *Lefschetz fixed point of* f if 1 is not an eigenvalue of $df_x : T_x(X) \to T_x(X)$. The map f is *Lefschetz* if all its fixed points are Lefschetz fixed points. Show that if X is compact and f is Lefschetz, then f has only finitely many fixed points.
(Hint: consider $F : X \to X \times X$ defined as $F(x) = (x, f(x))$, $x \in X$, and consider $\Delta(X) \subseteq X \times X$.)

(2) (a) (3 points) Given a manifold $X \subseteq \mathbb{R}^N$, define the *tangent space* $T_p(X)$ of X at a point $p \in X$.

(b) (8 points) Show that

$$X := \left\{(x_0, x_1, x_2, x_3)^t \in \mathbb{R}^4 : -x_0^2 + x_1^2 + x_2^2 + x_3^2 = -1 \text{ and } x_0 > 0\right\}$$

is a codimension 1 submanifold of $\mathbb{R}^4$. Explicitly identify its tangent space $T_x(X)$ for any $x \in X$.

(c) (9 points) Let M be the diagonal 4×4 matrix

$$M := \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and consider the set

$$O(1,3) := \{A \in \mathbb{R}^{4\times 4} : A^t M A = M\}.$$

This forms a group under matrix multiplication, i.e., $O(1,3)$ is closed under matrix multiplication, $O(1,3)$ contains the 4×4 identity matrix I_4, and every element in $O(1,3)$ has an inverse in $O(1,3)$ (you are not required to verify this).

Show that $O(1,3)$ is a 6-dimensional submanifold of $\mathbb{R}^{4\times 4}$, and that $O(1,3)$ is a Lie group. Explicitly identify the tangent space $T_{I_4}(O(1,3))$.

(d) (5 points) Recall the definitions of X and $O(1,3)$ in the above subquestions (b) and (c). Consider the subset

$$O^+(1,3) := \{A \in O(1,3) : Ax \in X \text{ for all } x \in X\}.$$

This is a subgroup of $O(1,3)$ (you are not required to verify this). Show that this is a submanifold of $O(1,3)$ of dimension 6.

(Hint: if $A \in O(1,3)$ and $x \in X$, then either $Ax \in X$ or $-Ax \in X$. It may also help to observe that if $x = (x_0, x_1, x_2, x_3)^t$, then $-x_0^2 + x_1^2 + x_2^2 + x_3^2 = x^t M x$.)

(3) Let X and Y be manifolds, and let $f : X \to Y$ be a smooth map.

(a) (7 points) Define what it means for $x \in X$ to be a regular point of f, and what it means for $y \in Y$ to be a regular value of f. Show that the set of regular points of f is open in X. State Sard's theorem.

(b) (8 points) In addition, show that if X is compact, then the set of regular values of f is open in Y. If, in addition, X has the same dimension as Y, show that whenever $y \in Y$ is a regular value of f, the preimage $f^{-1}(y)$ is a finite subset of X.

(c) (5 points) If $g : X \to Y$ is another smooth map, define what it means for f and g to be smoothly homotopic as maps from X to Y. Assuming again that X is compact and has the same dimension as Y, and assuming f and g are homotopic, show that for almost every point $y \in Y$, the preimages $f^{-1}(y)$ and $g^{-1}(y)$ are finite sets whose cardinalities have the same parity, i.e.,

$$|f^{-1}(y)| = |g^{-1}(y)| \mod 2.$$

(d) (5 points) Let n be a positive integer and let $U \subseteq \mathbb{S}^n$ be a nonempty open subset of $\mathbb{S}^n$. Suppose $h : \mathbb{S}^n \to \mathbb{S}^n$ is a smooth map such that for every $y \in U$, $h^{-1}(y)$ is a finite set whose cardinality is odd. Show that h must map some pair of antipodal points of $\mathbb{S}^n$ to another pair of antipodal points, i.e., there must exist a point $p \in \mathbb{S}^n$ such that $h(-p) = -h(p)$.

(4) (a) (3 points) State the Borsuk–Ulam theorem.

(b) (7 points) Prove the continuous version of the Borsuk–Ulam theorem: given a positive integer k and a continuous function $g : \mathbb{S}^k \to \mathbb{R}^k$, there exists a point $p \in \mathbb{S}^k$ such that $g(p) = g(-p)$.

(c) (8 points) Prove the covering variant of the Borsuk–Ulam theorem: given a positive integer k and given open subsets $U_1, \cdots, U_k$ of $\mathbb{S}^k$, there exists a point $p \in \mathbb{S}^k$ such that either

$$\exists i \in \{1, \cdots, k\} \quad (p \in U_i \text{ and } -p \in U_i),$$

or

$$\forall i \in \{1, \cdots, k\} \quad (p \notin U_i \text{ and } -p \notin U_i).$$

(Hint: given an open subset $U \subsetneq \mathbb{S}^k$, the function $g : \mathbb{S}^k \to \mathbb{R}$ defined by $g(x) = d(x, \mathbb{S}^k \setminus U) = \inf\{|x - y| : y \in \mathbb{S}^k \setminus U\}$ is continuous and takes nonzero values precisely on U.)

(d) (7 points) Let n and k be positive integers. Consider the set

$$V \coloneqq \{u \subseteq \{1, \cdots, 2n + k\} : |u| = n\},$$

where $|u|$ denotes the cardinality of u. Prove that for any disjoint partition of V into $k + 1$ sets $A_1, \cdots, A_{k+1}$, i.e.,

$$A_i \cap A_j = \varnothing \quad \text{if } 1 \leqslant i < j \leqslant k + 1$$

and

$$A_1 \cup \cdots \cup A_{k+1} = V,$$

there exists $i \in \{1, \cdots, k + 1\}$ and $u, v \in A_i$ such that $u \cap v = \varnothing$. (Hint: you may assume that there exist distinct points $p_1, \cdots, p_{2n+k} \in \mathbb{S}^{k+1}$ such that no $k+2$ of them lie on a great k-sphere, i.e., a set of the form $\{x \in \mathbb{S}^{k+1} : a \cdot x = 0\}$ for some $a \in \mathbb{S}^{k+1}$. Identify the numbers $1, \cdots, 2n + k$ with the points $p_1, \cdots, p_{2n+k}$. For each $i = 1, \cdots, k + 1$, define the set

$$U_i \coloneqq \{a \in \mathbb{S}^{k+1} : \exists u \in A_i \ \forall p \in u \ a \cdot p > 0\}.$$

Show U_i is open and apply part (c).)