FINAL ASSIGNMENT

MA 109A: INTRODUCTION TO GEOMETRY AND TOPOLOGY

Due on Friday December 12, 2025
(Last modified December 7, 2025)

(60 points in total)
Solve the following problems.

(1) (10 points) Let (X, d) be a compact metric space. Let
$$I(X) = \{f \in C(X, X) : d(x, y) = d(f(x), f(y)),\ f \text{ bijective}\}$$
be the set of isometries of X equipped with the compact-open topology.
 (a) Prove that a sequence $\{\phi_n\}_{n=1}^{\infty} \subset I(X)$ converges to $\phi \in X$ if and only if $\phi_n \to \phi$ pointwise on X as $n \to \infty$. (Hint: use Ascoli's theorem for the converse.)
 (b) Prove that $I(X)$ has a countable basis.
 (c) Prove that $I(X)$ is metrizable.
 (d) Prove that $I(X)$ with composition as the binary operation is a topological group.[1]

[1] For example, if $X = \mathbb{S}^{n-1}$, then $I(X) = O(n)$.

(2) (10 points) **Solve two of the following three subproblems.**

(a) Prove or disprove: the torus $\mathbb{S}^1 \times \mathbb{S}^1$ is a countable union of circles. More precisely, we put

$$\mathbb{S}^1 \times \mathbb{S}^1 = \{(e^{\mathrm{i}\phi}, e^{\mathrm{i}\psi}) : \phi, \psi \in \mathbb{R}\},$$

and circles are subsets of the form

$$\{(e^{\mathrm{i}(n\theta+\phi_0)}, e^{\mathrm{i}(m\theta+\psi_0)}) : \theta \in \mathbb{R}\},$$

where $\phi_0, \psi_0 \in \mathbb{R}$ are fixed and $(n, m) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ are integers, not both zero.

(b) Prove that an infinite dimensional Banach space does not have a countably infinite Hamel basis. More precisely, let X be a Banach space, i.e., a vector space with a norm under which it is a complete metric space. A subset $\{v_n\}_{n=1}^{\infty}$ of X is a (countably infinite) *Hamel basis* if it is a basis in the traditional sense: for every $x \in X$ there exists a unique sequence of real numbers $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}$ such that $a_n = 0$ for large enough n, i.e., $\exists N\ \forall n > N\ a_n = 0$, and $x = \sum_{n=1}^{N} a_n v_n$.

(c) Let f be an infinitely differentiable real-valued function on the real line $\mathbb{R}$. Suppose that for every $x \in \mathbb{R}$, there exists a nonnegative integer n_x such that the n_x-th derivative of f at x vanishes, i.e., $f^{(n_x)}(x) = 0$. Prove that f is a polynomial.

(3) (10 points)

Consider the following three different ways to prove that the fundamental group of a topological group is abelian (the third method will need some extra assumptions). That is, let G be a topological group with binary operation $\cdot$ and identity e_G. We will show $\pi_1(G, e_G)$ is abelian. **Choose and solve one of the following three strategies (a), (b), or (c).**

(a) Let f_1 and f_2 be two loops at e_G, i.e., we have two continuous maps $f_1, f_2 : [0,1] \to G$ such that $f_1(0) = f_1(1) = f_2(0) = f_2(1) = e_G$. Define $F : [0,1] \times [0,1] \to G$ as

$$F(s,t) = f_1(s) \cdot f_2(t).$$

Find two paths $\gamma_1, \gamma_2 : [0,1] \to [0,1] \times [0,1]$ with common endpoints such that $F \circ \gamma_1 = f_1 * f_2$ and $F \circ \gamma_2 = f_2 * f_1$. Show that γ_1 and γ_2 are homotopic, and argue why this implies $f_1 * f_2$ and $f_2 * f_1$ are homotopic.

(b) Consider the following result from algebra.[2]

Theorem 1 (Eckmann-Hilton argument)**.** *Let X be a set equipped with two binary operations, written $\circ$ and $\otimes$, such that:*

- *$\circ$ and $\otimes$ are both unital, i.e., there are elements $1_\circ$ and $1_\otimes$ such that $1_\circ \circ a = a \circ 1_\circ = a$ and $1_\otimes \otimes a = a \otimes 1_\otimes = a$ for all $a \in X$, and*
- *$(a \otimes b) \circ (c \otimes d) = (a \circ c) \otimes (b \circ d)$ for all $a, b, c, d \in X$.*

Then $\circ$ and $\otimes$ are the same and are associative and commutative.

On $\pi_1(G, e_G)$, consider the binary operation

$$[f] \otimes [g] = [s \mapsto f(s) \cdot g(s),\ s \in [0,1]], \quad [f], [g] \in \pi_1(G, e_G).$$

Show that this operation $\otimes$ is well-defined. Recall the ordinary group operation

$$[f] \circ [g] = [f * g], \quad [f], [g] \in \pi_1(G, e_G)$$

where $*$ denotes concatenation of paths. **Verify the assumptions of the Eckmann-Hilton argument for $\circ$ and $\otimes$,** which shows that $\circ$ is a commutative binary operation.

(c) (i) Carry out Exercise 6 of Section 79 on page 483 from the textbook *Topology, second edition* by J. Munkres.

(ii) Assume that (G, e_G) is a path connected, locally path connected, and semilocally simply connected topological group.[3] Take a universal cover $(\tilde{G}, e_{\tilde{G}})$ of G and a covering map $p : \tilde{G} \to G$

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[2] For those curious about the proof, we first observe that the units of the two operations coincide:

$$1_\circ = 1_\circ \circ 1_\circ = (1_\otimes \otimes 1_\circ) \circ (1_\circ \otimes 1_\otimes) = (1_\otimes \circ 1_\circ) \otimes (1_\circ \circ 1_\otimes) = 1_\otimes \otimes 1_\otimes = 1_\otimes$$

We then observe that the two operations coincide and are commutative:

$$a \circ b = (1 \otimes a) \circ (b \otimes 1) = (1 \circ b) \otimes (a \circ 1) = b \otimes a = (b \circ 1) \otimes (1 \circ a) = (b \otimes 1) \circ (1 \otimes a) = b \circ a$$

And finally we observe associativity:

$$(a \otimes b) \otimes c = (a \otimes b) \otimes (1 \otimes c) = (a \otimes 1) \otimes (b \otimes c) = a \otimes (b \otimes c)$$

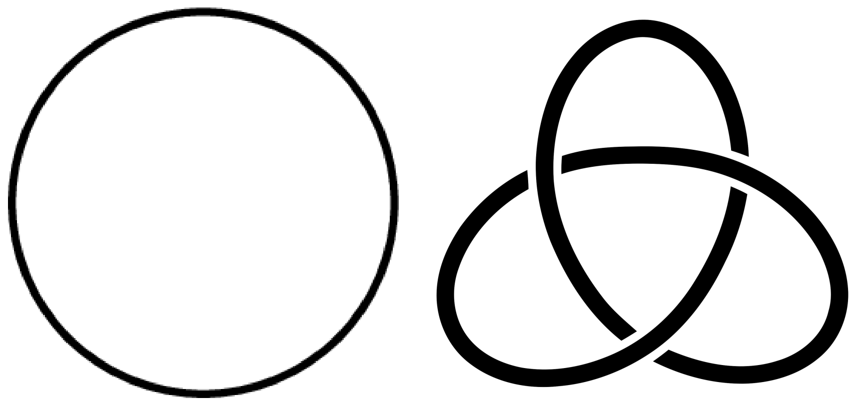
[3] This holds when G is in addition a manifold. In this case, G is called a *Lie group*.

mapping $\tilde{G}$ to G. By part (i), we may assume $\tilde{G}$ is a topological group with identity $e_{\tilde{G}}$ with p being a group homomorphism. Consider $N = p^{-1}(e_G)$, which is the kernel of p. This is a discrete normal subgroup of $\tilde{G}$. **Show that N with the group operation inherited from $\tilde{G}$ is isomorphic to $\pi_1(G, e_G)$, via the lifting correspondence (defined on page 345 of Munkres).** Then, **show that N is abelian,** by considering the map $G \to N : g \mapsto ghg^{-1}$ where h is a fixed element of N.

(4) (10 points) Consider the free group F_2 on two generators a and b. We have seen in class that any subgroup of F_2 is also free. In particular, the *commutator subgroup* $[F_2, F_2]$, which is the subgroup of F_2 generated by the commutators $ghg^{-1}h^{-1}$, $g, h \in F_2$, is also free. **Show that** $[F_2, F_2]$ **is not finitely generated.** Hint: consider B to be the wedge of two circles, and E to be the square edge-grid on $\mathbb{Z}^2$, i.e., the vertices are $\mathbb{Z}^2$ and two vertices are connected if one entry coincides and the other entry differs by 1. (Mathematically, $E = (\mathbb{Z} \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{Z})$.) Construct a suitable covering map $p : E \to B$.

(5) (10 points)

Let's do some knot theory. A *knot* is an embedding of the circle $\mathbb{S}^1$ into $\mathbb{R}^3$. For example, the *unknot* is the image of the map $\mathbb{S}^1 \to \mathbb{R}^3$ given by $e^{\mathrm{i}t} \mapsto (\cos t, \sin t, 0)$. Another example is the *trefoil*, which is the map $\mathbb{S}^1 \to \mathbb{R}^3$ given by $e^{\mathrm{i}t} \mapsto (\sin t + 2\sin 2t, \cos t - 2\cos 2t, -\sin 3t)$.



(A) The unknot (B) The trefoil

Two knots K and K' are equivalent if they are *isotopic*, i.e., there is a homotopy $H : \mathbb{S}^1 \times [0,1] \to \mathbb{R}^3$ such that $H_0 = K$, $H_1 = K'$, and for each $t \in [0,1]$, H_t is itself a knot. For a knot K, its *knot group* is $\pi_1(\mathbb{R}^3 \setminus K(\mathbb{S}^1))$. It is known that equivalent knots have isomorphic knot groups.

(a) Compute the knot group of the unknot.[4]

(b) It is known that the knot group of the trefoil is the finitely presented group

$$\left\langle a, b | a^2 = b^3 \right\rangle .$$

Prove that the unknot is not equivalent to the trefoil.

(c) What is the homology group of the knot group of the trefoil?

[4]Hint: Prove that by Seifert-van Kampen, it is equivalent to consider $\mathbb{S}^3 \setminus \mathbb{S}^1$, where $\mathbb{S}^1$ is some great circle. Justify why by stereographic projection this is homeomorphic to $\mathbb{R}^3 \setminus$ (some axis), and show that this deformation retracts onto $\mathbb{R}^2 \setminus \{pt\}$.

(6) (10 points)

State your favorite statement you learned in this course so far. Explain briefly about the statement as if you are giving a presentation in front of other students in mathematics who has little background in topology.

Alternatively, tell me a real-life observation/application of topology. One of mine is given in Figure 2.

(A) Hmmm... I wonder what they sell? (B) They were selling covers and knots!

FIGURE 2. A topologie store in Kyoto