MIDTERM

MA 109A: INTRODUCTION TO GEOMETRY AND TOPOLOGY

Questions

(1) (25 points in total)
   (a) (5 points) Let X be a topological space. Show that X is Hausdorff if and only if the diagonal $\Delta = \{(x,x) : x \in X\}$ is closed in $X \times X$ (given the product topology).
   (b) (5 points) Let X, Y be topological spaces, and let Y be Hausdorff. Given a continuous map $f : X \to Y$, show that its *graph*

$$\operatorname{Graph}(f) := \{(x, f(x)) : x \in X\} \subset X \times Y$$

   is closed in $X \times Y$ (given the product topology).
   (c) (8 points) Show that the converse to (b) holds with the additional assumption that Y is compact. Namely, if X, Y are topological spaces with Y compact Hausdorff, show that if $\operatorname{Graph}(f)$ is closed then f must be continuous.[1] Show that this fails without the compactness assumption for Y.
   (d) (7 points) Suppose $f, g : X \to Y$ are continuous maps and Y is Hausdorff. Show that

$$\{x \in X : f(x) = g(x)\}$$

   is closed in X. Give a counterexample if Y is not Hausdorff.

[1] Hint: first show that the projection map $\pi_1 : X \times Y \to X$ is closed if Y is compact.

(2) (15 points in total) In this problem, we will show that there is no nontrivial factorization $\mathbb{R} = X \times Y$, i.e., that if $\mathbb{R}$ is given the ordinary topology and X, Y are topological spaces such that $\mathbb{R}$ is homeomorphic to the product space $X \times Y$, then X is a point and Y is homeomorphic to $\mathbb{R}$, or vice versa.

(a) (5 points) Show that if X and Y are topological spaces such that the product space $X \times Y$ is connected, then X and Y must be connected.
(b) (5 points) Show that if X and Y are connected topological spaces, each with more than one point, then for any $x \in X$ and $y \in Y$, $X \times Y \smallsetminus \{x\} \times \{y\}$ must be connected.
(c) (5 points) Conclude that if X and Y are topological spaces such that $X \times Y$ is homeomorphic to $\mathbb{R}$, then either X is a singleton and Y is homeomorphic to $\mathbb{R}$, or vice versa.

(3) (20 points in total)

(a) (5 points) **Write the definition of a topological group.**

(b) (5 points) An *absolute value* on the rational numbers is a function $|\cdot|_* : \mathbb{Q} \to \mathbb{R}$ such that

(i) $|x|_* \geqslant 0$, with $|x|_* = 0$ if and only if $x = 0$,

(ii) $|x+y|_* \leqslant |x|_* + |y|_*$, and

(iii) $|xy|_* = |x|_*|y|_*$.

It is easily seen from these axioms that $|1|_* = |-1|_* = 1$ (you are not required to prove this).

Given an absolute value on $\mathbb{Q}$, **show that** $d_* : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$**,** $d_*(x, y) = |x - y|_*$**, is a metric on** $\mathbb{Q}$**, and that the additive group** $(\mathbb{Q}, +)$ **topologized by** d_* **is a topological group.**

(c) (5 points) Here are some examples of absolute values.

(i) There is the *trivial absolute value* $|\cdot|_0$ defined as

$$|x|_0 := \begin{cases} 0 & x = 0, \\ 1 & x \neq 0. \end{cases}$$

This induces a distance $d_0(x, y) = |x - y|_0$.

(ii) There is the *real absolute value* $|\cdot|_\infty$ defined as

$$|x|_\infty := \begin{cases} x & x \geqslant 0, \\ -x & x < 0. \end{cases}$$

This induces a distance $d_\infty(x, y) = |x - y|_\infty$.

(iii) For each prime number p, there is the *p-adic absolute value* $|\cdot|_p$ defined as

$$|x|_p := \begin{cases} 0 & x = 0, \\ p^{-n} & x = p^n \frac{a}{b},\ a \text{ and } b \text{ coprime integers not divisible by } p. \end{cases}$$

This induces a distance $d_p(x, y) = |x - y|_p$.

A theorem of Ostrowski (1916) states that these are essentially all the absolute values on $\mathbb{Q}$.

Theorem 1. *Let $|\cdot|_*$ be an absolute value on $\mathbb{Q}$. Then $|\cdot|_* = |\cdot|_q^\lambda$, where q is either 0, ∞, or a prime number, and $\lambda > 0$ is a positive real number.*

Show that the topological group $(\mathbb{Q}, +)$ **with distance** d_0 **has the discrete topology. Show that the topological group** $(\mathbb{Q}, +)$ **with distance** d_∞ **has the subspace topology induced by the inclusion** $\mathbb{Q} \subset \mathbb{R}$**, where** $\mathbb{R}$ **is given the ordinary topology.**

(d) (5 points) The following theorem was proven by Sierpinski in 1920.

Theorem 2. *A perfect countable metric space is homeomorphic to $(\mathbb{Q}, d_\infty)$.*

Here, a topological space X is said to be *perfect* if it has no isolated points, i.e., every point $x \in X$ belongs to the topological closure of its complement $X \smallsetminus \{x\}$.

Assuming Sierpinski's theorem (Theorem 2), **show that for each prime** p**,** $(\mathbb{Q}, d_p)$ **is homeomorphic to** $(\mathbb{Q}, d_\infty)$**.**

(4) (20 points in total)

The middle-third Cantor set C is the subspace of $[0,1]$ (with the ordinary topology induced from $\mathbb{R}$) obtained by successively removing the middle third of each interval, that is,

$$C = [0,1] \setminus \bigcup_{n \geqslant 0} \bigcup_{k=0}^{3^n-1} \left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}} \right).$$

Equivalently, one may write

$$C = \bigcap_{n \geqslant 0} C_n \quad C_0 = [0,1],\ C_n = \frac{1}{3} C_{n-1} \cup \left(\frac{2}{3} + \frac{1}{3} C_{n-1} \right).$$

(a) (3 points) **Show that C is nonempty and compact. Is C normal?**

Actually, one may represent points $x \in C$ uniquely as

$$x = \sum_{n=1}^{\infty} \frac{2a_n}{3^n}, \quad (a_1, a_2, a_3, \cdots) \in \{0,1\}^{\mathbb{Z}_{>0}}, \tag{1}$$

and we can also write, for each integer $N \geqslant 1$,

$$C = \bigsqcup_{a_1, \cdots, a_N \in \{0,1\}} \left(\sum_{n=1}^{N} \frac{2a_n}{3^n} + \frac{1}{3^N} C \right), \tag{2}$$

where $\bigsqcup$ means disjoint union. So this is a decomposition of C into disjoint compact subspaces.

(b) (3 points) Take the set $\{0,1\}$ with the discrete topology and form the infinite product $\{0,1\}^{\mathbb{Z}_{>0}}$ with the product topology $\mathcal{T}$. **Show that $(\{0,1\}^{\mathbb{Z}_{>0}}, \mathcal{T})$ is metrizable, with metric**

$$\mathbf{d}(\{\mathbf{x_n}\}_{\mathbf{n=1}}^{\infty}, \{\mathbf{y_n}\}_{\mathbf{n=1}}^{\infty}) = \sum_{\mathbf{n=1}}^{\infty} \mathbf{2^{-n}} |\mathbf{x_n} - \mathbf{y_n}|.$$

(c) (3 points) **Show that C is homeomorphic to $(\{0,1\}^{\mathbb{Z}_{>0}}, \mathcal{T})$ via the map $f : C \to \{0,1\}^{\mathbb{Z}_{>0}}$ given by**

$$\mathbf{f}\left(\sum_{\mathbf{n=1}}^{\infty} \frac{\mathbf{2a_n}}{\mathbf{3^n}} \right) = \{\mathbf{a_n}\}_{\mathbf{n=1}}^{\infty},$$

i.e., show that f is a homeomorphism (you can already assume that f is bijective, via the uniqueness of representation (1).[2])

(d) (3 points) **Show that the (visibly surjective) map $g : \{0,1\}^{\mathbb{Z}_{>0}} \to [0,1]$ given by**

$$\mathbf{g}\left(\{\mathbf{a_n}\}_{\mathbf{n=1}}^{\infty} \right) = \sum_{\mathbf{n=1}}^{\infty} \frac{\mathbf{a_n}}{\mathbf{2^n}}$$

is continuous.[3]

[2] Further hint: if you have difficulty showing that f is continuous, note the decomposition (2).

[3] Hint: The Weierstrass M-test should make this simple.

So one obtains a continuous surjection $g \circ f : C \to [0,1]$ given by

$$g \circ f \left(\sum_{n=1}^{\infty} \frac{2a_n}{3^n} \right) = \sum_{n=1}^{\infty} \frac{a_n}{2^n}.$$

(e) (2 points) **Is $g \circ f$ a homeomorphism?**

The Cantor set is not just some technical example made to bug you. It naturally appears quite often in the following sense:

Theorem 3 (Hausdorff-Alexandroff)**.** *Every compact metric space is a continuous image of the Cantor space C.*

(f) (3 points) Assuming the Hausdorff-Alexandroff Theorem 3, **show that every compact metric space is a quotient space of Cantor space C.**

Let's partially prove the Hausdorff-Alexandroff Theorem 3. Let (X, d) be a compact metric space. By replacing d with $\min\{d, 1\}$, we may assume d is valued in $[0,1]$. Now, X, being a compact metric space, is separable: it has a countable dense set $\{x_1, x_2, \cdots\}$. Consider the map $i : X \to [0,1]^{\mathbb{Z}_{>0}}$ defined by

$$i(x) = \{d(x, x_n)\}_{n=1}^{\infty}, \quad x \in X.$$

(g) (3 points) **Show that i is continuous and injective, and maps X homeomorphically onto its image in $[0,1]^{\mathbb{Z}_{>0}}$.**

The reminder of this page is a continuation of the proof of Theorem 3 and there is nothing I require you to prove.

On the other hand, from the homeomorphism $C \cong \{0,1\}^{\mathbb{Z}_{>0}}$, we have

$$C \cong \{0,1\}^{\mathbb{Z}_{>0}} = \{0,1\}^{\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}} \cong (\{0,1\}^{\mathbb{Z}_{>0}})^{\mathbb{Z}_{>0}} \cong C^{\mathbb{Z}_{>0}},$$

and a continuous surjection $C \to [0,1]$ induces a continuous surjection $C \cong C^{\mathbb{Z}_{>0}} \to [0,1]^{\mathbb{Z}_{>0}}$. If we denote by K the preimage in C of $i(X)$, then $i(X) \cong X$ is the continuous image of K. But there is the following Lemma, which we will not prove here:

Lemma 1. *Let K be a closed subspace of Cantor space C. Then there is a continuous surjection $C \to K$.*

Assuming this Lemma is true, we have continuous surjections $C \to K \to i(X) \cong X$, proving the Hausdorff-Alexandroff Theorem 3.

(5) (20 points in total)

Define a topology on the integers $\mathbb{Z}$, called the *evenly spaced integer topology*, where a subset $U \subset \mathbb{Z}$ is open if it is either empty or it is a union of arithmetic sequences $S(a, b)$ for $a \in \mathbb{Z} \setminus \{0\}$ and $b \in \mathbb{Z}$, where

$$S(a, b) := \{an + b : n \in \mathbb{Z}\}.$$

(a) (2 points) Verify the axioms of a topology for the evenly spaced integer topology by showing that the sets $S(a, b)$, $a \in \mathbb{Z} \setminus \{0\}$, $b \in \mathbb{Z}$, form a basis for the evenly spaced integer topology.

(b) (2 points) Show that a nonempty open set in the evenly spaced integer topology is infinite.

(c) (2 points) Show that each basis element $S(a, b)$, $a \in \mathbb{Z} \setminus \{0\}$, $b \in \mathbb{Z}$, is also closed.[4] So these are "clopen" sets.

(d) (1 point) Show that

$$\mathbb{Z} \setminus \{-1, 1\} = \bigcup_{p \text{ prime}} S(p, 0).$$

(e) (2 points) Using what we have shown so far, prove the infinitude of primes, by assuming that there are only finitely many primes and deriving a contradiction.[5]

Is $\mathbb{Z}$ with the evenly spaced integer topology...[6]

(f) (1 point) Connected?

(g) (1 point) Totally disconnected?[7]

(h) (1 point) Hausdorff?

(i) (1 point) Regular?

(j) (1 point) First countable?

(k) (1 point) Second countable?[8]

(l) (1 point) Normal?

(m) (1 point) Metrizable?

(n) (1 point) Perfect?[9][10]

(o) (1 point) Compact?

(p) (1 point) Locally compact?

[4] Hint: In case you need to refresh your memory, for $a \in \mathbb{Z}_{>0}$, the residue classes modulo a are $0, \cdots, a-1$, i.e., dividing by a results in remainders 0 through $a-1$. In our language, $\mathbb{Z}$ is a disjoint union of $S(a, 0), \cdots, S(a, a-1)$.

[5] This is Furstenberg's proof of the infinitude of primes.

[6] Hint: approach the statements below in the order I wrote them. These should be short conceptual checks.

[7] A topological space is said to be totally disconnected if the only nonempty connected subspaces are the singletons.

[8] Hint: what is a base for this topology?

[9] A topological space X is said to be *perfect* if it has no isolated points, i.e., every point $x \in X$ belongs to the topological closure of its complement $X \setminus \{x\}$.

[10] Now might be a good time to go back to Problem 3 and look at Theorem 2.