## PROBLEM SET #1

MA 109A: INTRODUCTION TO GEOMETRY AND TOPOLOGY

**Due on Friday October 10, 2025**
(Last modified October 3, 2025)

Solve the following problems. Alternatively, solve the optional problem announced during class.

(30 points in total)

- (5 points) A map $f : X \to Y$ between topological spaces is said to be *open* if for every open set $U$ of $X$, the image $f(U)$ is open in $Y$. For topological spaces $X$ and $Y$, show that the projection maps $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are open.
- (5 points) Let $X$ be a topological space. Show that $X$ is Hausdorff if and only if the diagonal $\Delta = \{(x, x) : x \in X\}$ is closed in $X \times X$ (given the product topology).
- (5 points) Let $X, Y$ be topological spaces, and let $Y$ be Hausdorff. Given a continuous map $f : X \to Y$, show that its *graph*
$$\mathrm{Graph}(f) := \{(x, f(x)) : x \in X\} \subset X \times Y$$
is closed in $X \times Y$ (given the product topology). Does the converse hold, i.e., if $\mathrm{Graph}(f)$ is closed, must $f$ be continuous?
- (5 points) Let $X, Y$ be topological spaces, let $A \subset X$ be a subspace, let $Y$ be Hausdorff, and let $f : A \to Y$ be continuous. Show that continuous extensions of $f$ to a function $\overline{A} \to Y$ are unique, i.e., if $g, h : \overline{A} \to Y$ are continuous functions such that $g|_A = h|_A = f$, then $g = h$.
- (10 points) Let $A$ be a set (not yet a topological space), let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of topological spaces, and let $\{f_\alpha\}_{\alpha \in J}$ be an indexed family of functions $f_\alpha : A \to X_\alpha$ (again, not yet continuous, as we haven't given $A$ a topology yet). Endow $A$ with the (unique) coarsest topology $\mathcal{T}$ such that every map $f_\alpha$, $\alpha \in J$, is continuous.[1]
  - (5 points) Given a topological space $Y$ and a map $g : Y \to A$, show that $g : Y \to A$ is continuous relative to $\mathcal{T}$ if and only if each composition $f_\alpha \circ g$, $\alpha \in J$, is continuous.
  - (5 points) Suppose the family $\{f_\alpha\}_{\alpha \in J}$ separates points, i.e., for any distinct $a, b \in A$ there is some $\alpha \in J$ such that $f_\alpha(a) \neq f_\alpha(b)$. Show that the map $f : A \to \prod_{\alpha \in J} X_\alpha$ given by $f(a) = (f_\alpha(a))_{\alpha \in J}$ is a topological embedding.

**Relevant course material for this homework:** Sections 12 - 19 of Munkres.

You may not search the solutions to the homework problems from any source. Please do not consult Large Language Models and please write the homework in your own words, as that is the best way to learn. If you collaborated with other students, please list their names on the solutions.

[1] In the setting of functional analysis, such a topology is called the *weak topology*.