PROBLEM SET #2

MA 109A: INTRODUCTION TO GEOMETRY AND TOPOLOGY

Due on Friday October 17, 2025
(Last modified October 10, 2025)

Solve the following problems. I've decided to add some problems that are (almost surely) not in Munkres, to give some excitement to this course. Please don't be intimidated by the length of this problem set, as they admit rather short solutions. Since this problem set is rather verbose, I will **bold** the parts that ask you to do something. If you are overwhelmed, please reach out to me at sryoo@caltech.edu.

(35 points in total)

(1) (5 points) Let $p : X \to Y$ be a continuous map. A *section* of p is a continuous map $f : Y \to X$ such that $p \circ f = \mathrm{id}_Y$. **Show that** if p has a section, then p is a quotient map. For example, given a subspace $A \subset X$, a *retraction* of X onto A is a continuous map $r : X \to A$ such that $r(a) = a$ for $a \in A$; **show that** a retraction admits a section and hence is a quotient map.

(2) (5 points) A T_1 topological space X is said to be *normal* if for every disjoint pair F, G of closed sets in X, there are disjoint open sets $U \supset F$ and $V \supset G$. **Show that** metric spaces are normal.

(3) (5 points) A T_1 topological space X is said to be *regular* if for any closed set F in X and a point $x \in X \setminus F$, there are disjoint open sets $U \ni x$ and $V \supset G$. **Show that** topological groups are regular.

(more problems on the next page)

(4) (10 points) Let's define uniformities. (There are two things I ask you to verify in this part.)

Definition 1. *A* relation *on a set X is a subset $U \subset X \times X$. For relations U, V on X, define the relations*

$$U^{-1} = \{(y,x) : (x,y) \in U\}, \quad U \circ V = \{(x,z) : (x,y) \in U,\ (y,z) \in V\}.$$

Recall the identity relation $\Delta = \Delta(X) = \{(x,x) : x \in X\}$. *For a relation U on X and an element $x \in X$, define*

$$U[x] = \{y \in X : (x,y) \in U\}.$$

A uniformity *on X is a nonempty family $\mathcal{U}$ of relations on X such that*
(a) if $U \in \mathcal{U}$ then $\Delta \subset U$,
(b) if $U \in \mathcal{U}$ then $U^{-1} \in \mathcal{U}$,
(c) if $U \in \mathcal{U}$, then $V \circ V \subset U$ for some $V \in \mathcal{U}$,
(d) if $U, V \in \mathcal{U}$ then $U \cap V \in \mathcal{U}$,
(e) if $U \in \mathcal{U}$ and $U \subset V \subset X \times X$ then $V \in \mathcal{U}$.
The pair $(X, \mathcal{U})$ is called a uniform space.

Consider the collection

$$\mathcal{T}_{\mathcal{U}} = \{T \subset X : \forall x \in T \exists U \in \mathcal{U}\ U[x] \subset T\}.$$

Verify that $\mathcal{T}_{\mathcal{U}}$ defines a topology on X. This is called the *uniform topology* on X induced by the uniform space $(X, \mathcal{U})$.

Similarly to topological spaces, we may naturally consider bases (and also subbases).

Definition 2. *A subfamily $\mathcal{B}$ of a uniformity $\mathcal{U}$ is a* base *for $\mathcal{U}$ if each member of $\mathcal{U}$ contains a member of $\mathcal{B}$.*

Note that $\mathcal{B}$ determines $\mathcal{U}$, since $U \subset X \times X$ belongs to $\mathcal{U}$ iff U contains a member of $\mathcal{B}$. The following is not hard to verify (and is not required of you).

Theorem 1. *A nonempty family $\mathcal{B}$ of subsets of $X \times X$ is a base for some uniformity for X iff*
(a) $\forall B \in \mathcal{B}\ \Delta \subset B$
(b) $\forall B \in \mathcal{B}\ B^{-1} \in \mathcal{B}$
(c) $\forall B \in \mathcal{B}\ \exists C \in \mathcal{B}\ C \circ C \subset B$
(d) $\forall B, C \in \mathcal{B}\ \exists D \in \mathcal{B}\ D \subset B \cap C$.

Theorem 2. *Let $\mathcal{B}$ be a base for a uniformity $\mathcal{U}$. Then*

$$\{B[x] \subset X : B \in \mathcal{B}\}$$

is a base for the uniform topology induced by $\mathcal{U}$.

Using these results, **show that**[1] metric spaces and topological groups have uniform topologies induced by some uniformity.

If you wish to know more about uniformities, please consult Chapter 6 of the book *General Topology* by John Kelley.

[1] Please don't think too hard on this problem! A hint is that, by the above two theorems, you just need to specify what the base is.

(5) (10 points) Let's make an excursion into functional analysis. (There are three things I ask you to show.)

Definition 3. *Suppose $\mathcal{T}$ is a topology on a real vector space X such that*
(a) $\mathcal{T}$ is T_1, and
(b) the vector space operations, namely addition $+ : X \times X \to X$ and scalar multiplication $\cdot : \mathbb{R} \times X \to X$ are continuous.
Then $\mathcal{T}$ is called a vector topology *on X, and X with the topology $\mathcal{T}$ is called a* topological vector space.

Show that a topological vector space X is regular.[2]

Associate to each $a \in X$ and $\lambda \in \mathbb{R} \setminus \{0\}$ the translation operator $T_a : X \to X$ and multiplication operator $M_\lambda : X \to X$ defined by

$$T_a(x) = a + x,\ M_\lambda(x) = \lambda x, \quad \forall x \in X.$$

Show that T_a and M_λ are homeomorphisms of X onto X.

Definition 4. *Let N be a linear subspace of a vector space X. As in the case of groups, for every $x \in X$, let $\pi(x) = x + N$ be the coset of N that contains x. These cosets form a vector space X/N, called the* quotient space of X modulo N, *where addition and scalar multiplication are defined as*

$$\pi(x + y) = \pi(x) + \pi(y),\ \alpha\pi(x) = \pi(\alpha x), \quad x, y \in X,\ \alpha \in \mathbb{R}.$$

(These operations are well-defined.) Thus, $\pi : X \to X/N$ becomes a linear map with kernel N, called the quotient map *of X onto X/N.*

If $\mathcal{T}$ is a vector topology on X and N is a closed linear subspace of X, give X/N the quotient topology $\mathcal{T}_N$ induced by the map $\pi : X \to X/N$ is a quotient map. **Verify** that $\mathcal{T}_N$ is a vector topology on X/N and that the quotient map $\pi : X \to X/N$ is open and continuous.[3]

Relevant course material for this homework: Up to section 22 of Munkres.

You may not search the solutions to the homework problems from any source. Please do not consult Large Language Models and please write the homework in your own words, as that is the best way to learn. If you collaborated with other students, please list their names on the solutions.

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[2]Hint: X is an example of one of the concepts we have learned. Which one is it?

[3]Hint: you are allowed to assume problem 5 on page 146 of Munkres, as this was covered in class. To verify that $\mathcal{T}_N$ is a vector topology, stare at the following diagrams.

$$\begin{tikzcd} X \times X \arrow[r, "+"] \arrow[d, "\pi \times \pi"'] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "\pi"] \\ X/N \times X/N \arrow[r, dashed, "+"] & X/N \end{tikzcd} \qquad \begin{tikzcd} \mathbb{R} \times X \arrow[r, "\cdot"] \arrow[d, "\mathrm{id}_{\mathbb{R}} \times \pi"'] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "\pi"] \\ \mathbb{R} \times X/N \arrow[r, dashed, "\cdot"] & X/N \end{tikzcd}$$