PROBLEM SET #3

MA 109A: INTRODUCTION TO GEOMETRY AND TOPOLOGY

(1) (5 points) Fix $n \geq 2$. Show that $(0,1)$, $(0,1]$, $[0,1]$, and $(0,1)^n$ are all pairwise not homeomorphic.[1]

(2) (5 points) Show that any continuous function $f : [0,1] \to [0,1]$ has a fixed point, i.e., there is a point $x \in [0,1]$ such that $f(x) = x$.[2]

(3) (5 points) Show that if A is a countable subset of $\mathbb{R}^2$, then $\mathbb{R}^2 \setminus A$ is connected.

(4) (5 points) Recall from Problem Set 1 that if $f : X \to Y$ is a continuous map between topological spaces, with Y Hausdorff, then its graph

$$\mathrm{Graph}(f) := \{(x, f(x)) : x \in X\} \subset X \times Y$$

is closed in $X \times Y$, but the converse fails in general, i.e., $\mathrm{Graph}(f)$ can be closed without f being continuous. **Show that** the converse holds with the additional assumption that Y is compact.[3]

(5) (7 points) Let (X, d) be a metric space and let $f : X \to X$ be a map.
   (a) The map f is called a *contraction* if there is a number $\alpha < 1$ such that

   $$d(f(x), f(y)) \leq \alpha d(x, y), \quad x, y \in X.$$

   Show that if X is complete and $f : X \to X$ is a contraction, then f has a fixed point.[4]

   (b) The map f is called a *shrinking map* if

   $$d(f(x), f(y)) < d(x, y), \quad \text{for all } x, y \in X \text{ with } x \neq y.$$

   Show that if X is compact and $f : X \to X$ is a shrinking map, then f has a fixed point.

(more problems on the next page)

[1] I am not asking you to show, for example, that $(0,1)^2$ and $(0,1)^3$ are not homeomorphic (which is true, for reference).

[2] In general, for any $n \geq 1$, any continuous function $f : [0,1]^n \to [0,1]^n$ has a fixed point; this is called the *Brouwer fixed point theorem*.

[3] Hint: first show that the projection map $\pi_1 : X \times Y \to X$ is closed if Y is compact.

[4] This is called the Banach fixed point theorem.

(6) (8 points) Let X be a set. A *bornology* on X is a collection $\mathcal{B}$ of subsets of X such that
  (a) $\mathcal{B}$ covers X: $\cup\mathcal{B} = X$,
  (b) $\mathcal{B}$ is downward closed: if $B \in \mathcal{B}$ and $A \subset B$, then $A \in \mathcal{B}$, and
  (c) $\mathcal{B}$ is closed under finite unions, i.e., if $B_1, \cdots, B_n \in \mathcal{B}$, then $\cup_{i=1}^{n} B_i \in \mathcal{B}$.

The idea behind a bornology is that it is a notion of "bounded sets". If X is a T_1 topological space, **show that** the set $\mathcal{B}$ of *precompact* subsets of X is a bornology, where $A \subset X$ is said to be precompact if $\overline{A}$ is compact.

If G is a topological group, a bornology $\mathcal{B}$ on G is a *group bornology* if in addition to (a)-(c) we have
  (d) For $A, B \in \mathcal{B}$, $A \cdot B := \{ab : a \in A, b \in B\} \in \mathcal{B}$ and $A^{-1} := \{a^{-1} : a \in A\} \in \mathcal{B}$.

Show that the set of precompact subsets of G form a group bornology.

If X is a topological vector space,[5] a bornology $\mathcal{B}$ on X is a *vector bornology* if in addition to (a)-(c) we have
  (d') For $A, B \in \mathcal{B}$ and $\lambda \in \mathbb{R} \setminus \{0\}$, $A + B := \{a + b : a \in A, b \in B\} \in \mathcal{B}$ and $\lambda A := \{\lambda a : a \in A\} \in \mathcal{B}$.

Show that the set of precompact subsets of X form a vector bornology.[6]

[5]See problem set 2 for the definition of a topological vector space.

[6]You are allowed to assume results from the previous problem sets.

(7) (Optional, 10 points) Let p be a prime. A *p-adic integer* is a sequence in $\{0, \cdots, p-1\}$, i.e.,

$$(a_0, a_1, a_2, \cdots), \quad a_i \in \{0, \cdots, p-1\}.$$

which we also write in p-ary form

$$\overline{\cdots a_2 a_1 a_0}.$$

Addition and multiplication are done just as with integers, i.e.,

$$\overline{\cdots a_2 a_1 a_0} + \overline{\cdots b_2 b_1 b_0} = \overline{\cdots c_2 c_1 c_0},$$

where $c_0 \equiv a_0 + b_0 \mod p$, and if $a_0 + b_0 = c_0 + p$, we carry over a 1, and so on. The collection of all p-adic integers is denoted $\mathbb{Z}_p$. This isn't a group theory class, so let's take for granted that all the usual rules of algebra hold (so that $\mathbb{Z}_p$ is a ring). We won't be using the product structure of $\mathbb{Z}_p$ in this exercise, so you can view $\mathbb{Z}_p$ as an additive group.

Similarly to integers, we may define a norm on $\mathbb{Z}_p$, though its properties are quite different. The p-adic norm $|x|_p$ of $x = \overline{\cdots a_2 a_1 a_0}$ is defined as 0 if $\overline{\cdots a_2 a_1 a_0} = \overline{\cdots 000}$, and otherwise it is defined as p^{-k}, where k is the smallest integer so that $a_k \neq 0$. It satisfies the relations

(a) $|x|_p \geq 0$ for all $x \in \mathbb{Z}_p$, with equality iff $x = 0$.
(b) $|xy|_p = |x|_p |y|_p$,
(c) $|x + y|_p \leq \max\{|x|_p, |y|_p\} \leq |x|_p + |y|_p$,[7]

which defines a metric on $\mathbb{Z}_p$ by $d(x, y) = |x - y|_p$.

Being a metric space, $\mathbb{Z}_p$ is Hausdorff (and in fact normal). Moreover, $(\mathbb{Z}_p, +)$ is known to be an example of a *totally disconnected locally compact group*[8]. **Show that $(\mathbb{Z}_p, +)$ is totally disconnected and compact** through the following outline.

(a) First observe that the balls in $\mathbb{Z}_p$ are of the form

$$\begin{aligned} B(x, r) &= \{y : |x - y|_p < r\} = \{y : |x - y|_p \leq p^{-\nu}\} \\ &= \{\overline{\cdots y_2 y_1 y_0} : y_i = x_i \text{ for } 0 \leq i < \nu\}, \end{aligned}$$

where ν is the smallest $\nu \geq 0$ such that $p^{-\nu} < r$.[9] **Show that if X is a metric space where open balls are closed, then X is totally disconnected**, where a topological space is said to be *totally disconnected* if the only connected subspaces are the singletons.

(b) Consider the map $\phi : \{0, \cdots, p-1\}^{\mathbb{Z}_{\geq 0}} \to \mathbb{Z}_p$ where $\phi(a_0, a_1, a_2, \cdots) = \overline{\cdots a_2 a_1 a_0}$. **Show that ϕ is continuous,** where $\{0, \cdots, p-1\}$ is given the discrete topology and $\{0, \cdots, p-1\}^{\mathbb{Z}_{\geq 0}}$ is given the product topology. By the Tychonoff theorem, $\{0, \cdots, p-1\}^{\mathbb{Z}_{\geq 0}}$ is compact, being the product of compact spaces. Therefore the image of ϕ, $\mathbb{Z}_p$, is compact. Actually, ϕ is a homeomorphism, so $\mathbb{Z}_p$ is homeomorphic

[7] A metric space with this stronger triangle inequality is called an *ultrametric* space.

[8] tdlc group, for short. In fact the field of fractions, $\mathbb{Q}_p$, namely the *p-adic numbers*, consist of either the zero element, or sequences of the form

$$\overline{\cdots a_{-n+2} a_{-n+1} a_{-n}}$$

for some integer n where $a_{-n} \neq 0$, which is defined to have norm p^n. The additive group $\mathbb{Q}_p$ is also a tdlc group.

[9] In fancy words, the norm $|\cdot|_p$ is a *discrete valuation.*

to $\{0, \cdots, p-1\}^{\mathbb{Z}_{\geq 0}}$, which actually turns out to be homeomorphic to the Cantor set.

Totally disconnected locally compact groups are an active area of research; see, for example, `https://zerodimensional.group/reading_group/190227_michal_ferov.pdf` for an exposition.

Relevant course material for this homework: Up to section 29 of Munkres.

You may not search the solutions to the homework problems from any source. Please do not consult Large Language Models and please write the homework in your own words, as that is the best way to learn. If you collaborated with other students, please list their names on the solutions.