PROBLEM SET #5

MA 109A: INTRODUCTION TO GEOMETRY AND TOPOLOGY

Due on Friday November 14, 2025
(Last modified November 7, 2025)

(30 points in total, 10 bonus points)

(1) (5 points) Let X be a completely regular space. Show that X is connected if and only if its Stone-Čech compactification $\beta(X)$ is connected.[1]

(2) (5 points) Let Y be an arbitrary compactification of the completely regular space X, i.e., Y is a compact Hausdorff space containing X as a subspace such that $\overline{X} = Y$. Denote by $\beta(X)$ the Stone-Čech compactification of X. Show that there is a continuous surjective closed map $g : \beta(X) \to Y$ equalling the identity on X.

Thus $\beta(X)$ is the "maximal compactification of X," in the sense that every compactification of X is a quotient space of $\beta(X)$.

(3) (5 points) Show that the product space $[0, 1]^{[0,1]}$ is separable. Show that $\beta(\mathbb{Z}_+)$ has cardinality at least as great as $[0, 1]^{[0,1]}$.

[1]Hint: For one direction, consult Theorem 23.4. For the other direction, let $X = A \cup B$ be a separation of X, so that the function $f : X \to [0.1]$ defined by

$$f(x) = \begin{cases} 0, & x \in A, \\ 1, & x \in B, \end{cases}$$

is continuous. What does this extend to in $\beta(X)$?

(4) (a) (5 points) A topological space X is said to be *σ-compact* if there is a sequence of compact subspaces $\{K_n\}_{n=1}^{\infty}$ such that $X = \bigcup_{n=1}^{\infty} K_n$. Let's call a σ-compact space X *nested σ-compact*[2] if we can take the compact subspaces to satisfy in addition $K_n \subset \mathrm{int}(K_{n+1})$, $n \in \mathbb{Z}_{>0}$. Show that Hausdorff, nested σ-compact spaces are paracompact. This generalizes the fact that $\mathbb{R}^n$ is paracompact, and actually has almost the same proof verbatim (see Example 1 on page 253).

(b) (3 points) Show that locally compact, connected topological groups are paracompact.[3]

(c) (2 points) Show that open subsets of $\mathbb{R}^n$ are nested σ-compact.

(d) (3 points) Recall that an m-manifold M is a second countable Hausdorff space that is locally homeomorphic to an open subset of $\mathbb{R}^m$. Show that m-manifolds are paracompact.[4]

(e) (2 points) Show that manifolds are metrizable.[5][6]

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[2] This is just an arbitrary terminology I came up with; it probably isn't widely used.

[3] Hint: let U_1 be a neighborhood of the identity with compact closure. Define $U_{n+1} = \overline{U_n} \cdot U_1$ recursively. Observe that $\bigcup_{n=1}^{\infty}$ is clopen, and that the sets $\overline{U_n}$ form a nested sequence of compact sets.

[4] This will be relevant when learning about manifolds (such as in Ma109c). For example, Theorem 41.7 tells you that any indexed open covering of a manifold admits a dominated partition of unity. Of course, when you learn about *smooth* manifolds, you will have to revisit the proofs to obtain a *smooth* partition of unity, which would also require us to obtain a smooth version of Urysohn's lemma; more on this in Ma109c.

[5] Hint: Smirnov metrization theorem. This problem generalizes the last problem of the previous problem set. As mentioned there, if the manifold is smooth, you can actually metrize the manifolds using a Riemannian metric.

[6] In fact, the converse is true. If M is a connected, locally Euclidean, Hausdorff space, then M is second countable iff M is σ-compact iff M is paracompact iff M is metrizable. See "A comprehensive introduction to differential geometry Vol. 1" by Michael Spivak, p.459.

(5) (Optional, 10 points) Let's look at an application of Tychonoff's theorem in functional analysis.

(a) (3 points) Let X be a topological vector space. Denote by X^* its *continuous dual space*, i.e., the space of continuous linear maps $x^* : X \to \mathbb{R}$. Given $x^*, y^* \in X^*$ and $a \in \mathbb{R}$, we can define $x^* + y^*, ax^* \in X^*$ in the obvious way:

$$(x^* + y^*)(x) = x^*(x) + y^*(x), \quad (ax^*)(x) = a(x^*(x)), \quad x \in X.$$

It is clear that these are continuous, so they are indeed members of X^*.

On the dual space X^* we can introduce the weak-* topology, or $\sigma(X^*, X)$-topology,[7] where the basis of neighborhoods of a point $x_0^* \in X^*$ is given by

$$U(x_0; \varepsilon, x_1, \cdots, x_n) = \{x^* \in X^* : |x^*(x_j) - x_0^*(x_j)| < \varepsilon \text{ for } j = 1, \cdots, n\}.$$

Verify that this defines a vector topology on X^*.

(b) (2 points) If $(X, \|\cdot\|)$ is a Banach space, endowed with the topology induced by its norm $\|\cdot\|$, one can characterize the dual as the space of functions $x^* : X \to \mathbb{R}$ for which

$$\|x^*\|_* := \sup_{x \in X : \|x\| \leq 1} |x^*(x)| < \infty.$$

It is clear that $\|\cdot\|_*$ turns X^* into a normed space. **Verify that** X^* is a Banach space.

(c) (5 points) Given a Banach space X, define its closed unit ball

$$B_X = \{x \in X : \|x\| \leq 1\}.$$

It is known that if X is infinite dimensional and is endowed with the topology induced by its norm $\|\cdot\|$, then B_X is not compact.[8] Likewise, in this case X^* is also infinite dimensional, and B_{X^*} is not compact in the topology induced by $\|\cdot\|_*$. However, it is compact in the weak-* topology:

Theorem 1 (Banach-Alaoglu)**.** *The closed unit ball B_{X^*} of X^* is $\sigma(X^*, X)$-compact.*

Let's prove this theorem. Define a map $\Phi : B_{X^*} \to P = \prod_{x \in X} [-\|x\|, \|x\|]$ by $\Phi(x^*) = \{x^*(x)\}_{x \in X}$. **Show that P with the product topology is compact**. By definition of the $\sigma(X^*, X)$-topology and product topology, and the last problem of Problem Set 1, Φ is a homeomorphic embedding. To end the proof, **show that $\Phi(B_{X^*})$ is closed in P**, from the identity

$$\Phi(B_{X^*}) = \{\{k_x\}_{x \in X} \in P : k_{\lambda x} = \lambda k_x,\ k_{x+y} = k_x + k_y,\ \forall x, y \in X,\ \lambda \in \mathbb{R}\}.$$

Relevant course material for this homework: Up to section 45 of Munkres.

You may not search the solutions to the homework problems from any source. Please do not consult Large Language Models and please write the homework in your own words, as that is the best way to learn. If you collaborated with other students, please list their names on the solutions.

[7] Analogously, one may define a weak topology, or $\sigma(X, X^*)$-topology, on X.

[8] Even in the weak topology, B_X is generally not compact. A fact is that the unit ball B_X is weakly compact iff X is *reflexive*, which I won't define in this course.