**FINAL EXAMINATION**

MA 109C: INTRODUCTION TO GEOMETRY AND TOPOLOGY

# Questions

(1) (25 points)
  (a) (6 points) State the definition of an orientation of a finite-dimensional vector space, and the definition of an orientation of a manifold. (Don't forget the 0-dimensional case!)
  (b) (6 points) For a non-negative integer $p$, state the definition of an alternating $p$-tensor on a vector space, and the definition of a $p$-form on a manifold.
  (c) (6 points) State the (generalized) Stokes' theorem.
  (d) (7 points) Define the vector fields $X$, $Y$ on $\mathbb{R}^3$ as follows:

$$X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y} - \frac{\partial}{\partial z}.$$

  Compute $[X, Y]$. Do these form an integrable distribution?

(2) (25 points)

(a) (6 points) State the definition of a smooth vector field $v$ on a manifold $X$, and the definition of the index of $v$ at an isolated zero $x \in X$ of $v$.

(b) (6 points) Let $X$ be a compact oriented manifold without boundary. State the definition of the Euler characteristic $\chi(X)$ of $X$. State the Poincaré–Hopf theorem for $X$.

(c) (7 points) Prove that for positive integers $k \geqslant 1$,

$$\chi(\mathbb{S}^k) = \begin{cases} 2, & k \text{ even} \\ 0, & k \text{ odd.} \end{cases}$$

(d) (6 points) What is $\chi(SO(8))$, where $SO(8)$ is the special orthogonal group consisting of the $8 \times 8$ real orthogonal matrices of determinant 1?

(3) (25 points) In this problem, we will prove that there exist real numbers $x, y, z, w \in \mathbb{R}$ such that

$$\begin{cases} x^3 - 3xy^2 - 3xz^2 - 3xw^2 - 3625x^2 - 121 = 0, \\ 3x^2y - y^3 - yz^2 - yw^2 + 6495x - 74625w = 0, \\ 3x^2z - y^2z - z^3 - zw^2 + 143xy = 0, \\ 3x^2w - y^2w - z^2w - w^3 - 10 = 0. \end{cases} \tag{1}$$

The strategy of the proof will be to imitate the proof of the Fundamental Theorem of Algebra.

(a) (5 points) State the Fundamental Theorem of Algebra.

Following the proof of the Fundamental Theorem of Algebra, write the above system (1) of equations as

$$F(x, y, z, w) + G(x, y, z, w) = 0,$$

where $F, G : \mathbb{R}^4 \to \mathbb{R}^4$ are the polynomials such that each component of $F$ is homogeneous of degree 3 and each component of $G$ has degree less than 3. It follows from a computation, which you are not required to show, that

$$\begin{aligned} &(x^3 - 3xy^2 - 3xz^2 - 3xw^2)^2 + (3x^2y - y^3 - yz^2 - yw^2)^2 \\ &+ (3x^2z - y^2z - z^3 - zw^2)^2 + (3x^2w - y^2w - z^2w - w^3)^2 \\ &= (x^2 + y^2 + z^2 + w^2)^3, \quad (x, y, z, w) \in \mathbb{R}^4, \end{aligned}$$

or equivalently,

$$|F(x, y, z, w)| = |(x, y, z, w)|^3, \quad (x, y, z, w) \in \mathbb{R}^4,$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^4$.

(b) (5 points) Show that there exists some $R > 0$ such that

$$|F(x, y, z, w)| > |G(x, y, z, w)|$$

whenever $(x, y, z, w) \in \mathbb{R}^4$ is such that $|(x, y, z, w)| = R$.

Now, assume for contradiction that there are no solutions to (1) in $\mathbb{R}^4$.

(c) (8 points) For $R > 0$ given as in (b), show that the map $F/|F| : \partial B_R(0) \to \mathbb{S}^3$ is well-defined and has degree zero.

(d) (7 points) Use the Borsuk–Ulam theorem on $F/|F| : \partial B_R(0) \to \mathbb{S}^3$ to deduce that the mod 2 degree of $F/|F|$ is 1. Show that its degree must be odd.

The contradiction between (c) and (d) tells us that (1) must have a solution, as we set out to prove.

(4) (25 points)

(a) (5 points) Let $m$ be a positive integer. Consider the unit circle $\mathbb{S}^1$ as a subset of $\mathbb{C}$, given the boundary orientation as the boundary of the unit disc $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leqslant 1\}$. Define the map $f : S^1 \to S^1$ as $f(z) = z^m$. Show that $1 \in S^1$ is a regular value of $f$ with exactly $m$ preimages, each point at which the differential $df$ is orientation preserving. From this deduce that the degree of $f$ is $m$.

(b) (7 points) Now, considering $\mathbb{S}^1$ as a subset of $\mathbb{R}^2$ with coordinates $(x, y)$, consider the differential 1-form $\omega = -ydx + xdy$ (more precisely, the 1-form $-ydx + xdy$ is defined on $\mathbb{R}^2$, and we pull this back to $\mathbb{S}^1$ via the inclusion map). Compute the integral

$$\int_{\mathbb{S}^1} \omega = 2\pi$$

and the pullback

$$f^*\omega = m\omega.$$

Thus

$$\int_{\mathbb{S}^1} f^*\omega = m \int_{\mathbb{S}^1} \omega \quad \text{and} \quad \int_{\mathbb{S}^1} \omega \neq 0.$$

Explain why this and the degree formula gives a different proof that the degree of $f$ is m.

(Hint: for computing the pullback, note that if we define the vector field $v(x, y) = (-y, x)$ on $\mathbb{S}^1$, then $\frac{d}{dz} z^m = mz^{m-1}$ for $z$ a complex variable implies that $df(v) = mv$. Now observe that $\omega(v) = 1$.)

(c) (5 points) Now consider the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$, given the boundary orientation from the closed unit ball. Consider the differential 2-form on $\mathbb{S}^2$ given by $\nu = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$ (more precisely, this 2-form is defined on $\mathbb{R}^3$, and we pull this back to $\mathbb{S}^2$ via the inclusion map). Compute the integral

$$\int_{\mathbb{S}^2} \nu$$

(Hint: This is the area of the unit sphere.)

(d) (8 points) Define the map $g : \mathbb{S}^2 \to \mathbb{S}^2$ given by

$$g(x, y, z) = \left( \frac{x^2 - y^2}{1 + z^2}, \frac{2xy}{1 + z^2}, \frac{2z}{1 + z^2} \right), \quad (x, y, z) \in \mathbb{S}^2.$$

This map is well-defined and smooth (you are not required to check this). Compute the pullback

$$g^*\nu = \frac{4(1 - z^2)}{(1 + z^2)^3} \left( 2zdx \wedge dy + xdy \wedge dz + ydz \wedge dx \right).$$

and the integral

$$\int_{\mathbb{S}^2} g^*\nu$$

Explain why this and the degree formula shows that $\deg(g) = 2$.[1]

[1] For reference, the map $g$ is the map induced by the square map $z \mapsto z^2 : \mathbb{C} \to \mathbb{C}$ via stereographic projection. In general, any rational function $P(z)/Q(z)$ induces a smooth map from the Riemann sphere $\mathbb{S}^2$ to itself, and the degree of that map is $\deg P - \deg Q$. You are not required to show any of the statements in this footnote.

Hint: Once you compute $g^*\nu$, computing the integral $\int_{\mathbb{S}^2} g^*\nu$ can proceed in two ways. You can either use the fact that

$$\int_{\mathbb{S}^2} (adx \wedge dy + bdy \wedge dz + cdz \wedge dx) = \int_{\mathbb{S}^2} (ax + by + cz)dS,$$

where $dS$ denotes the area element on $\mathbb{S}^2$, or you could use Stokes' theorem. Either way, you will likely have to use the fact that

$$\frac{d}{dz}\left(\frac{z}{1+z^2}\right) = \frac{1-z^2}{(1+z^2)^2}.$$