

MIDTERM

MA 109C: INTRODUCTION TO GEOMETRY AND TOPOLOGY

Questions

- (1) (30 points in total)
- (a) (10 points) Given sets X and Y a function $f : X \rightarrow Y$, the *graph* of f is defined as

$$\text{graph}(f) := \{(x, f(x)) \in X \times Y : x \in X\}.$$

Define the function $F : X \rightarrow \text{graph}(f)$ by

$$F(x) = (x, f(x)), \quad x \in X.$$

Show that if X and Y are manifolds and f is smooth, then $\text{graph}(f)$ is a manifold and $F : X \rightarrow \text{graph}(f)$ is a diffeomorphism. Show that F has derivative

$$dF_x(v) = (v, df_x(v)), \quad x \in X, \quad v \in T_x(X),$$

when we view $\text{graph}(f)$ as a submanifold of the product manifold $X \times Y$. Show that

$$T_{(x, f(x))}(\text{graph}(f)) = \text{graph}(df_x : T_x(X) \rightarrow T_{f(x)}(Y)).$$

- (b) (5 points) Given a set X , the diagonal $\Delta(X)$ is defined as

$$\Delta(X) := \{(x, x) \in X \times X : x \in X\}.$$

Show that if X is a manifold, then $\Delta(X)$ is a manifold of the same dimension as X . Show that

$$T_{(x, x)}(\Delta(X)) = \Delta(T_x(X)).$$

- (c) (5 points) Let V be a finite dimensional real vector space, and let $A : V \rightarrow V$ be a linear map. Show that, as submanifolds of $V \times V$, $\text{graph}(A) \cap \Delta(V)$ if and only if 1 is not an eigenvalue of A .
- (d) (10 points) Let X be a manifold and let $f : X \rightarrow X$ be a smooth map. A fixed point $x \in X$ of f , i.e., a point $x \in X$ such that $f(x) = x$, is a *Lefschetz fixed point* of f if 1 is not an eigenvalue of $df_x : T_x(X) \rightarrow T_x(X)$. The map f is *Lefschetz* if all its fixed points are Lefschetz fixed points. Show that if X is compact and f is Lefschetz, then f has only finitely many fixed points.
(Hint: consider $F : X \rightarrow X \times X$ defined as $F(x) = (x, f(x))$, $x \in X$, and consider $\Delta(X) \subseteq X \times X$.)

- (2) (30 points in total)
- (a) (5 points) Given a manifold $X \subseteq \mathbb{R}^N$, define the *tangent space* $T_p(X)$ of X at a point $p \in X$.
- (b) (10 points) Show that
- $$X := \{(x_0, x_1, x_2, x_3)^t \in \mathbb{R}^4 : -x_0^2 + x_1^2 + x_2^2 + x_3^2 = -1 \text{ and } x_0 > 0\}$$
- is a codimension 1 submanifold of $\mathbb{R}^4$. Explicitly identify its tangent space $T_x(X)$ for any $x \in X$.
- (c) (10 points) Let M be the diagonal 4×4 matrix

$$M := \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and consider the set

$$O(1, 3) := \{A \in \mathbb{R}^{4 \times 4} : A^t M A = M\}.$$

This forms a group under matrix multiplication, i.e., $O(1, 3)$ is closed under matrix multiplication, $O(1, 3)$ contains the 4×4 identity matrix I_4 , and every element in $O(1, 3)$ has an inverse in $O(1, 3)$ (you are not required to verify this).

Show that $O(1, 3)$ is a 6-dimensional submanifold of $\mathbb{R}^{4 \times 4}$, and that $O(1, 3)$ is a Lie group. Explicitly identify the tangent space $T_{I_4}(O(1, 3))$.

- (d) (5 points) Recall the definitions of X and $O(1, 3)$ in the above subquestions (b) and (c). Consider the subset

$$O^+(1, 3) := \{A \in O(1, 3) : Ax \in X \text{ for all } x \in X\}.$$

This is a subgroup of $O(1, 3)$ (you are not required to verify this). Show that this is a submanifold of $O(1, 3)$ of dimension 6.

(Hint: if $A \in O(1, 3)$ and $x \in X$, then either $Ax \in X$ or $-Ax \in X$. It may also help to observe that if $x = (x_0, x_1, x_2, x_3)^t$, then $-x_0^2 + x_1^2 + x_2^2 + x_3^2 = x^t M x$.)

- (e) (Bonus: 10 points) Show that $O(1, 3)$ has four connected components. (Hint: show that $O^+(1, 3) \cap SO(1, 3)$ is a connected subgroup of $O(1, 3)$ of index 4, where $SO(1, 3) = \{A \in O(1, 3) : \det A = 1\}$.)

(3) (40 points) Let X and Y be manifolds, and let $f : X \rightarrow Y$ be a smooth map.

- (a) (10 points) Define what it means for $x \in X$ to be a regular point of f , and what it means for $y \in Y$ to be a regular value of f . Show that the set of regular points of f is open in X . State Sard's theorem.
- (b) (10 points) In addition, show that if X is compact, then the set of regular values of f is open in Y . If, in addition, X has the same dimension as Y , show that whenever $y \in Y$ is a regular value of f , the preimage $f^{-1}(y)$ is a finite subset of X .
- (c) (10 points) If $g : X \rightarrow Y$ is another smooth map, define what it means for f and g to be smoothly homotopic as maps from X to Y . Assuming again that X is compact and has the same dimension as Y , and assuming f and g are homotopic, show that for almost every point $y \in Y$, the preimages $f^{-1}(y)$ and $g^{-1}(y)$ are finite sets whose cardinalities have the same parity, i.e.,

$$|f^{-1}(y)| = |g^{-1}(y)| \pmod{2}.$$

- (d) (10 points) Let n be a positive integer and let $U \subseteq \mathbb{S}^n$ be a nonempty open subset of $\mathbb{S}^n$. Suppose $h : \mathbb{S}^n \rightarrow \mathbb{S}^n$ is a smooth map such that for every $y \in U$, $h^{-1}(y)$ is a finite set whose cardinality is odd. Show that h must map some pair of antipodal points of $\mathbb{S}^n$ to another pair of antipodal points, i.e., there must exist a point $p \in \mathbb{S}^n$ such that $h(-p) = -h(p)$.