PROBLEM SET #7

MA 109C: INTRODUCTION TO GEOMETRY AND TOPOLOGY

Due on Friday May 30, 2024

(1) Define the vector fields X, Y on $\mathbb{R}^3$ as follows:
$$X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y} - \frac{\partial}{\partial z}.$$
 (a) Find a new coordinate chart on a neighborhood U of $(0,0,0)$ with coordinate functions (x_1, x_2, x_3) such that $X = \frac{\partial}{\partial x_1}$ on U.
 (b) Compute $[X, Y]$.
 (c) Is there a coordinate chart on a neighborhood U of $(0,0,0)$ with coordinate functions (x_1, x_2, x_3) such that $X = \frac{\partial}{\partial x_1}$ and $Y = \frac{\partial}{\partial x_2}$ on U?

(2) Prove or disprove. Let X be a vector field on a smooth manifold M. For $f \in C^\infty(M)$, one has $Xf = 0$ if and only if f is constant on every integral curve for X.

(3) Prove or disprove. Let α be a 1-form on a smooth manifold M. If there exists a nowhere-vanishing function $f \in C^\infty(M)$ such that $d(f\alpha) = 0$, then $\alpha \wedge d\alpha = 0$ holds.

(4) Let $M = \{(x, y, z) \in \mathbb{R}^3 : x > 0, y > 0\}$. Consider the distribution $D \subset TM$ whose fiber is given by
$$D_{(x,y,z)} = \operatorname{span}\left\{ y\frac{\partial}{\partial x} + xy\frac{\partial}{\partial z}, x\frac{\partial}{\partial y} + xy\frac{\partial}{\partial z} \right\}.$$
 (a) Show that D is integrable.
 (b) Describe the integral manifolds for D.

Relevant course material for this homework: Up to the content covered on May 14th.

You may not search the solutions to the homework problems from any source and you must write the homework in your own words. If you collaborated with other students, you must list their names on the solutions.