PROBLEM SET #1

MA 157C: TOPICS IN GEOMETRY AND TOPOLOGY

Due on Friday April 11, 2025
(Last modified April 2, 2025)

(1) Recall the definition of a Lie subgroup: it is a subgroup that is an embedded submanifold. Let G be a Lie group and H a Lie subgroup.
   (a) Show that the closure $\overline{H}$ of H in G is a subgroup of G.
   (b) Show that each coset Hx, $x \in \overline{H}$, is dense in $\overline{H}$.
   (c) Show that $\overline{H} = H$, i.e., every Lie subgroup is closed.

(2) Show that every discrete normal subgroup N of a connected Lie group G is central. (Hint: consider the map $G \to N : g \mapsto ghg^{-1}$ for a fixed $h \in N$.) By applying this to the kernel of the covering map $\tilde{G} \to G$ of the universal cover, conclude that $\pi_1(G)$ is commutative.

(3) (a) Let $G_{n,k}$ be the Grassmannian, the set of all dimension k subspaces in $\mathbb{R}^n$. Show that $G_{n,k}$ is a homogeneous space for $O(n)$ and identify it with the coset space $O(n)/H$ for appropriate H. Use it to define on $G_{n,k}$ a manifold structure and find its dimension.
   (b) On $\mathbb{R}^{2n}$, let ω be the standard symplectic form $\omega(x, y) = \sum_{i=1}^{n} (x_i y_{i+n} - y_i x_{i+n})$. A subspace $V \leq \mathbb{R}^{2n}$ is *Lagrangian* if $\dim V = n$ and $\omega(x, y) = 0$ for all $x, y \in V$. Let L_n be the set of all Lagrangian subspaces of $(\mathbb{R}^{2n}, \omega)$. Show that $Sp(2n, \mathbb{R})$ acts transitively on L_n. Use it to define on L_n a manifold structure and find its dimension.

(4) (a) Using the long exact sequence of homotopy groups associated with fiber bundles, show that $\pi_1(SO(n+1)) = \pi_1(SO(n))$ for $n \geq 3$. Since $\pi_1(SO(3)) = \mathbb{Z}_2$, this implies $\pi_1(SO(n)) = \mathbb{Z}_2$ for $n \geq 3$.
   (b) Let $GL_+(n, \mathbb{R})$ denote the matrix Lie group of invertible $n \times n$ real matrices of positive determinant. Using the Gram-Schmidt orthogonalization process, show that $GL_+(n, \mathbb{R})/SO(n)$ is diffeomorphic to the space of upper-triangular matrices with positive entries on the diagonal. Deduce that $GL_+(n, \mathbb{R})$ is homotopy equivalent to $SO(n)$, and compute its fundamental group.