# PROBLEM SET #2

MA 157C: TOPICS IN GEOMETRY AND TOPOLOGY

**Due on Friday April 25, 2025**
(Last modified April 15, 2025)

(1) (a) Verify that $\mathbb{R}^3$, with the commutator given by the cross product, is a Lie algebra. Verify that it is isomorphic as a Lie algebra to $\mathfrak{so}(3, \mathbb{R})$.
   (b) Let $\varphi : \mathfrak{so}(3, \mathbb{R}) \to \mathbb{R}^3$ be the isomorphism of part (a). Prove that under this isomorphism, the standard action of $\mathfrak{so}(3, \mathbb{R})$ on $\mathbb{R}^3$ is identified with the action of $\mathbb{R}^3$ on itself given by the cross-product:
$$a \cdot \vec{v} = \varphi(a) \times \vec{v}, \quad a \in \mathfrak{so}(3, \mathbb{R}),\ \vec{v} \in \mathbb{R}^3.$$

(2) Let $P_n$ be the space of polynomials over $\mathbb{R}$ of degree at most $n$ in the variable $x$. The Lie group $G = \mathbb{R}$ acts on $P_n$ by translations: $\rho(t)(x) = x + t$, $t \in G$. Prove that the corresponding action of the Lie algebra $\mathfrak{g} = \mathbb{R}$ is given by $\rho(a) = a\partial_x$, $a \in \mathfrak{g}$ and deduce from this the Taylor formula for polynomials:
$$f(x + t) = \sum_{n \geq 0} \frac{(t\partial_x)^n f}{n!}.$$

(3) Recall the following basis of $\mathfrak{so}(3, \mathbb{R})$:
$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad J_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$
By abuse of notation, denote the induced vector fields on $\mathbb{R}^3$ via the action of $\mathfrak{so}(3, \mathbb{R})$ on $\mathbb{R}^3$:
$$J_x = -z\partial_y + y\partial_z, \quad J_y = -x\partial_z + z\partial_x, \quad J_z = -y\partial_x + x\partial_y.$$
Define the second-order differential operator on $\mathbb{R}^3$ given by $\Delta_{sph} = J_x^2 + J_y^2 + J_z^2$, called the spherical Laplace operator.
   (a) Express $\Delta_{sph}$ in terms of $x, y, z, \partial_x, \partial_y, \partial_z$.
   (b) Show that if $f$ is a function on $\mathbb{R}^3$, then $\Delta_{sph} f|_{\mathbb{S}^2}$ depends only on $f|_{\mathbb{S}^2}$. Thus $\Delta_{sph}$ is a well-defined differential operator on $\mathbb{S}^2$.
   (c) Show that if $f$ is a function on $\mathbb{S}^2$ and $T \in SO(3, \mathbb{R})$, then
$$\Delta_{sph}(f \circ T^{-1}) = (\Delta_{sph} f) \circ T^{-1}.$$
   Thus $\Delta_{sph}$ is rotation-invariant.
   (d) Show that the usual Laplace operator $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ can be written as
$$\Delta = \frac{1}{r^2}\partial_r \left(r^2 \partial_r\right) + \frac{1}{r^2}\Delta_{sph}.$$