PROBLEM SET #1

MA 109C: INTRODUCTION TO GEOMETRY AND TOPOLOGY

Due on Friday April 10, 2025
(Last modified April 2, 2025)

(Below, the exercises refer to the problems from the textbook *Differential Topology* by V. Guillemin and A. Pollack. Exercise $a.b.c$ means exercise c from section b of chapter a.)

Either:

(1) Solve the following exercises:
- Exercise 1.1.17
- Exercise 1.2.11
- Exercise 1.5.10

(It may help to notice that exercise 1.1.17 implies exercise 1.1.16, and exercise 1.2.11 implies exercise 1.2.10.)

(2) Or, solve the following problems:
- Exercise 1.3.2
- Determine whether the following statement is true. Let X and Z be submanifolds of the manifold Y. Denote $\dim X = k$, $\dim Z = l$, $\dim Y = n$, and assume $k + l \geq n$. If $X \pitchfork Z$, then for each $p \in X \cap Z$, there exists a local coordinate system $(x_1, \cdots, x_n)$ of a neighborhood U of p in Y so that

$$U \cap X = \{q \in U : x_{k+1}(q) = \cdots = x_n(q) = 0\},$$

$$U \cap Z = \{q \in U : x_1(q) = \cdots = x_{n-l}(q) = 0\}.$$

In words, the transversal intersection $X \pitchfork Z$ is locally equivalent to the "canonical transversal intersection"

$$\left(\mathbb{R}^k \times \{0\}^{n-k}\right) \pitchfork \left(\{0\}^{n-l} \times \mathbb{R}^l\right) \text{ in } \mathbb{R}^n.$$

(3) Or, solve the following problem:
- Recall our definition of smoothness: given a subset $X \subset \mathbb{R}^N$, a function $f : X \to \mathbb{R}^M$ is said to be *smooth* if for each $x \in X$ there is a neighborhood U_x of x in $\mathbb{R}^N$ and an infinitely differentiable function $F_x : U_x \to \mathbb{R}^M$ such that $F_x|_{U_x} = f|_{U_x}$. Let's make another notion of smoothness: a function $f : X \to \mathbb{R}^M$ is *globally smooth* if there is an open set U in $\mathbb{R}^N$ containing X and an infinitely differentiable function $F : U \to \mathbb{R}^M$ such that $F|_X = f$.

  Clearly, global smoothness implies smoothness. Does smoothness imply global smoothness? If you want a hint, shoot me an email at `sryoo@caltech.edu`.

You may not search the solutions to the homework problems from any source and you must write the homework in your own words. If you collaborated with other students, you must list their names on the solutions.

I strongly recommend that you solve, but not submit solutions for, the following additional problems, for self-study.

- Chapter 1, section 2, exercises 9, 12
- Chapter 1, section 3, exercises 9, 10
- Chapter 1, section 4, exercises 7, 11, 13
- Chapter 1, section 5, exercises 4, 5, 7