PROBLEM SET #2

MA 109C: INTRODUCTION TO GEOMETRY AND TOPOLOGY

Due on Friday April 17, 2026

Exercise numbers refer to the textbook *Differential Topology* by V. Guillemin and A. Pollack.

Solve either (1), (2), or (3) (note that they each carry optional bonus problems):

(1) explore properties of the sphere by solving the following problems:
- Exercises 1.6.7, 1.8.7, 1.8.8 (you may have a look at exercise 1.8.6)
- Construct nonvanishing tangent vector fields X_1, X_2, X_3 on S^3 such that X_1, X_2, X_3 are linearly independent.[1] Using this, show that the tangent bundle of S^3 is trivial in the sense described in class.
- (Optional problem, bonus 1 point) Construct nonvanishing tangent vector fields
$$X_1, X_2, X_3, X_4, X_5, X_6, X_7$$
on S^7 such that $X_1, X_2, X_3, X_4, X_5, X_6, X_7$ are linearly independent. Using this, show that the tangent bundle of S^7 is trivial.[2]

[1] Pssst! Quaternions!

[2] It is a deep fact that the tangent bundle of S^n is trivial if and only if $n = 0, 1, 3, 7$. It is a consequence of Adams' Hopf Invariant One Theorem.

(2) Explore Morse functions by solving:
- Exercises 1.7.16, 1.7.17, 1.7.18
- Let $N \geqslant 1$ be a natural number. For each $a_1, \cdots, a_N \in \mathbb{R}$, consider the system of equations

$$F_i^{a_1,\cdots,a_N}(\theta_1,\cdots,\theta_N) \coloneqq \frac{1}{N}\left(\sum_{j=1}^{N}\sin(\theta_j-\theta_i)-\sin\theta_i\right)-a_i=0, \quad i=1,\cdots,N \tag{1}$$

of the variables $\theta_1, \cdots, \theta_N \in [0, 2\pi]$. (This is the equation of equilibria for the *Kuramoto model*.) Show that the set of $(a_1, \cdots, a_N)$ such that the "potential function"

$$P^{a_1,\cdots,a_N}(\theta_1,\cdots,\theta_N)=\frac{1}{N}\left(\frac{1}{2}\sum_{i,j=1}^{N}\cos(\theta_i-\theta_j)+\sum_{i=1}^{N}\cos\theta_i\right)-\sum_{i=1}^{N}a_i\theta_i$$

is Morse is open and has full measure in $\mathbb{R}^N$. Conclude that the set

$$A=\{(a_1,\cdots,a_N)\in\mathbb{R}^N : (1) \text{ has finitely many solutions in } [0,2\pi]^N\}$$

has full measure[3] in $\mathbb{R}^N$ (i.e., $\mathbb{R}^N \setminus A$ has measure zero).
- (Optional, bonus 1 point) The above potential function $P^{a_1,\cdots,a_N}$ is defined on $\mathbb{R}^N$. However, the system of equations (1) finds itself more naturally situated on the torus $\mathbb{T}^N = (\mathbb{S}^1)^N$. A problem for the Kuramoto model is to construct *Lyapunov functionals*, namely smooth functions $L^{a_1,\cdots,a_N} : \mathbb{T}^N \to \mathbb{R}$ such that

$$\sum_{i=1}^{N}\frac{\partial L_{a_1,\cdots,a_N}}{\partial\theta_i}(\theta_1,\cdots,\theta_N)\cdot F_i^{a_1,\cdots,a_N}(\theta_1,\cdots,\theta_N)\geqslant 0, \tag{2}$$

with equality happening exactly at the solutions of (1).
Show that for some $N \geqslant 1$, for any $\varepsilon > 0$, there exist $|a_1|, \cdots, |a_N| \leqslant \varepsilon$ that do not admit Lyapunov functionals.[4]
- (Optional, bonus $+\infty$ points) Let B denote the set of $(a_1, \cdots, a_N) \in [-1/2, 1/2]^N$ for which there exists a Lyapunov functional as in (2). Show that B has full measure in $[-1/2, 1/2]^N$.

[3] Shameless plug: it is a theorem of myself and others that $A = \mathbb{R}^N \setminus \{0\}$, and that the number of solutions is bounded by 2^{N+1} on A. (Seung-Yeal Ha, Hwa Kil Kim, and Seung-Yeon Ryoo. "On the finiteness of collisions and phase-locked states for the Kuramoto model." Journal of Statistical Physics 163, no. 6 (2016): 1394-1424.)

[4] Hint: take $N = 3$, $a_1 = 0$, $a_2 = a_3 = \varepsilon$, and consider the configurations $\theta_1 = \pi$, $\theta_2 = \theta$, and $\theta_3 = \theta + \pi$. What does (2) say for these configurations?

(3) Explore properties of the Lorentz group by solving:
- Show that
$$X := \left\{(x_0, x_1, x_2, x_3)^t \in \mathbb{R}^4 : -x_0^2 + x_1^2 + x_2^2 + x_3^2 = -1 \text{ and } x_0 > 0\right\}$$
is a codimension 1 submanifold of $\mathbb{R}^4$. Explicitly identify its tangent space $T_x(X)$ for any $x \in X$.
- Let M be the diagonal 4×4 matrix
$$M := \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
and consider the set
$$O(1,3) := \{A \in \mathbb{R}^{4\times 4} : A^t M A = M\}.$$
This forms a group under matrix multiplication, i.e., $O(1,3)$ is closed under matrix multiplication, $O(1,3)$ contains the 4×4 identity matrix I_4, and every element in $O(1,3)$ has an inverse in $O(1,3)$ (you are not required to verify this). This is known as the *Lorentz group*, the symmetry group in special relativity.
Show that $O(1,3)$ is a 6-dimensional submanifold of $\mathbb{R}^{4\times 4}$ (so that $O(1,3)$ is a Lie group). Explicitly identify the tangent space $T_{I_4}(O(1,3))$.
- Consider the subset[5]
$$O^+(1,3) := \{A \in O(1,3) : Ax \in X \text{ for all } x \in X\}.$$
This is a subgroup of $O(1,3)$ (you are not required to verify this). Show that this is a submanifold of $O(1,3)$ of dimension 6.[6]
- (Optional problem, bonus 1 point) Show that $O(1,3)$ has four connected components.[7]

You may not search the solutions to the homework problems from any source and you must write the homework in your own words. If you collaborated with other students, you must list their names on the solutions.

[5] These are the transformations that preserve the direction of time, a.k.a. the *orthochronous* operators.

[6] Hint: if $A \in O(1,3)$ and $x \in X$, then either $Ax \in X$ or $-Ax \in X$. One may trivially observe that if $x = (x_0, x_1, x_2, x_3)^t$, then $-x_0^2 + x_1^2 + x_2^2 + x_3^2 = x^t M x$.

[7] Hint: show that $O^+(1,3) \cap SO(1,3)$ is a connected subgroup of $O(1,3)$ of index 4, where $SO(1,3) = \{A \in O(1,3) : \det A = 1\}$ is the collection of *proper* operators, i.e., those that preserve parity.

I strongly recommend that you solve, but not submit solutions for, the following additional problems from the textbook.

- Chapter 1, section 6, exercises 1, 2, 3, 4, 5, 6, 10
- Chapter 1, section 7, exercises 6, 11, 14, 22
- Chapter 1, section 8, exercises 3, 6