PROBLEM SET #3

MA 109C: INTRODUCTION TO GEOMETRY AND TOPOLOGY

Due on Friday April 24, 2026

For nonnegative integers n, k with $n \geqslant k$, the *Grassman manifold* is the set $G_{n,k}$ of k-dimensional linear subspaces (or k-planes, for short) of $\mathbb{R}^n$ with a certain manifold structure. Since we're defining manifolds to be subsets of $\mathbb{R}^N$, we need to bijectively embed $G_{n,k}$ into some Euclidean space and prove that the image has a manifold structure. Let's consider the bijective embedding

$$G_{n,k} \to \mathrm{Sym}(\mathbb{R}^{n\times n}), \quad V \mapsto P_V,$$

where P_V is the orthogonal projection onto V. By linear algebra, one can verify the image as

$$G_{n,k} = \{P \in \mathrm{Sym}(\mathbb{R}^{n\times n}) : P^2 = P,\ \mathbf{Tr}\, P = k\},$$

where we abuse notation and denote the image as $G_{n,k}$ as well.

- Verify that $G_{n,k}$ is a submanifold of $\mathrm{Sym}(\mathbb{R}^{n\times n}) \cong \mathbb{R}^{n(n+1)/2}$. Compute its dimension.

  Hint: let $F : \mathrm{Sym}(\mathbb{R}^{n\times n}) \to \mathrm{Sym}(\mathbb{R}^{n\times n})$, $F(P) = P^2 - P$. At $P \in G_{n,k}$, decompose any $H \in \mathrm{Sym}(\mathbb{R}^{n\times n})$ as $H = PHP + (I-P)H(I-P) + (\mathrm{off} - \mathrm{diag})$. Then

$$DF_P(PHP) = PHP,\ DF_P((I-P)H(I-P)) = -(I-P)H(I-P),\ DF_P(\mathrm{off} - \mathrm{diag}) = 0.$$

  So $\ker DF_P$ is just the off-diagonal part, of dimension [fill blank here]. The trace condition just selects connected components of $P^2 = P$. Finish by the preimage theorem.
- Prove that the map $G_{n,k} \to G_{n,n-k}$, $V \mapsto V^\perp$, is a diffeomorphism.
- Consider $G_{n,k}$ embedded in $G_{n+1,k+1}$ by identifying a k-plane $P \subseteq \mathbb{R}^n$ with $P \times \mathbb{R} \subseteq \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$. If $\dim M \leqslant k$, then every map $f : M \to G_{n+1,k+1}$ is homotopic to a map $g : M \to G_{n,k}$. If $\dim M < k$, then the homotopy class of g is uniquely determined by that of f, i.e., if $f : M \to G_{n+1,k+1}$ is homotopic to $f' : M \to G_{n+1,k+1}$, $f : M \to G_{n+1,k+1}$ is homotopic to $g : M \to G_{n,k}$, and $f' : M \to G_{n+1,k+1}$ is homotopic to $g' : M \to G_{n,k}$, then $g, g' : M \to G_{n,k}$ are homotopic as maps $M \to G_{n,k}$.
- The *real projective n-space* $\mathbb{R}P^n$ is simply $G_{n+1,1}$. It can be thought of as the equivalence class $[x_0, \cdots, x_n]$ of $(n+1)$-tuples of real numbers not all zero, where the equivalence relation is $[x_0, \cdots, x_n] = [\lambda x_0, \cdots, \lambda x_n]$ where λ is a nonzero real number. It can also be thought of as the quotient of $\mathbb{S}^n$ with respect to the relation $x \sim -x$.

  Embeddings of $\mathbb{R}P^n$ into S^{n+k} for certain values of k can be constructed as follows. Suppose we had a symmetric bilinear map $h : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+k+1}$ such that $h(x, y) \neq 0$ if $x \neq 0$ and $y \neq 0$. Define $g : \mathbb{S}^n \to \mathbb{S}^{n+k}$ by $g(x) = h(x, x)/|h(x, x)|$.

- Show that $g(x) = g(y)$ if and only if $x = \pm y$. (Hint: consider $h(x + \lambda y, x - \lambda y)$ if $h(x,x) = \lambda^2 h(y,y)$.
- Show that g induces an analytic embedding $\mathbb{R}P^n \to S^{n+k}$.
- Consider the map $h : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{2n+1}$,

  $$h(x_0, \cdots, x_n, y_0, \cdots, y_n) = \left(\sum_{i+j=k} x_i y_j \right)_{k=0}^{2n}.$$

  Show that $\mathbb{R}P^n$ embeds in $\mathbb{S}^{2n}$ for all n.
- Prove the Whitney embedding theorem for $\mathbb{R}P^n$, i.e., show $\mathbb{R}P^n$ embeds in $\mathbb{R}^{2n}$.

You may not search the solutions to the homework problems from any source and you must write the homework in your own words. If you collaborated with other students, you must list their names on the solutions.